HIGGS PRODUCTION AT N3LO

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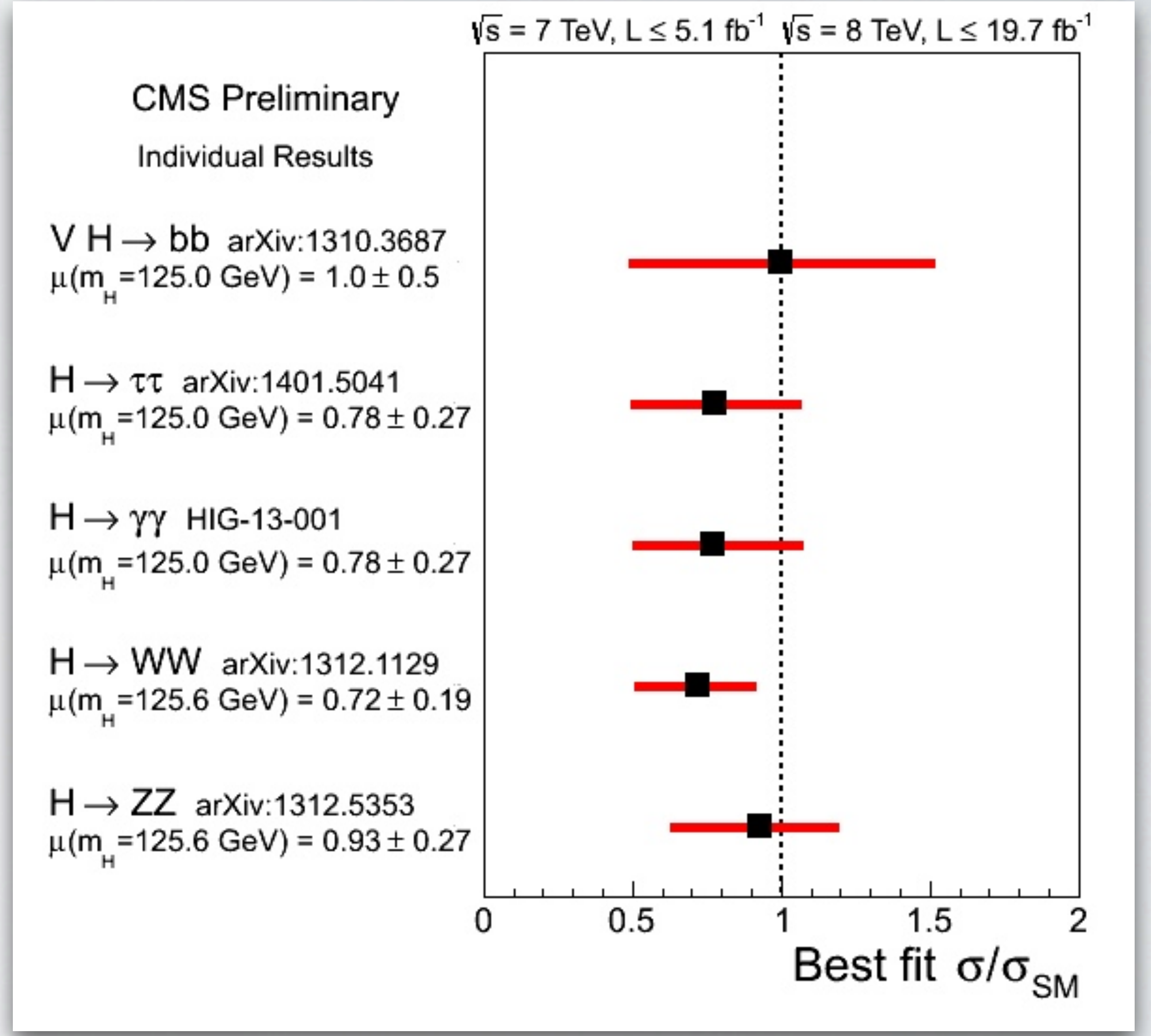
- Why do we compute?
- What do we want to compute?
- How do we compute it?
- What do we find?

Motivation

- Discovery marks the beginning of the experimental era of Higgs physics

- Determination of the properties of the Higgs will be a challenge for years to come

- Requires precision measurements and predictions



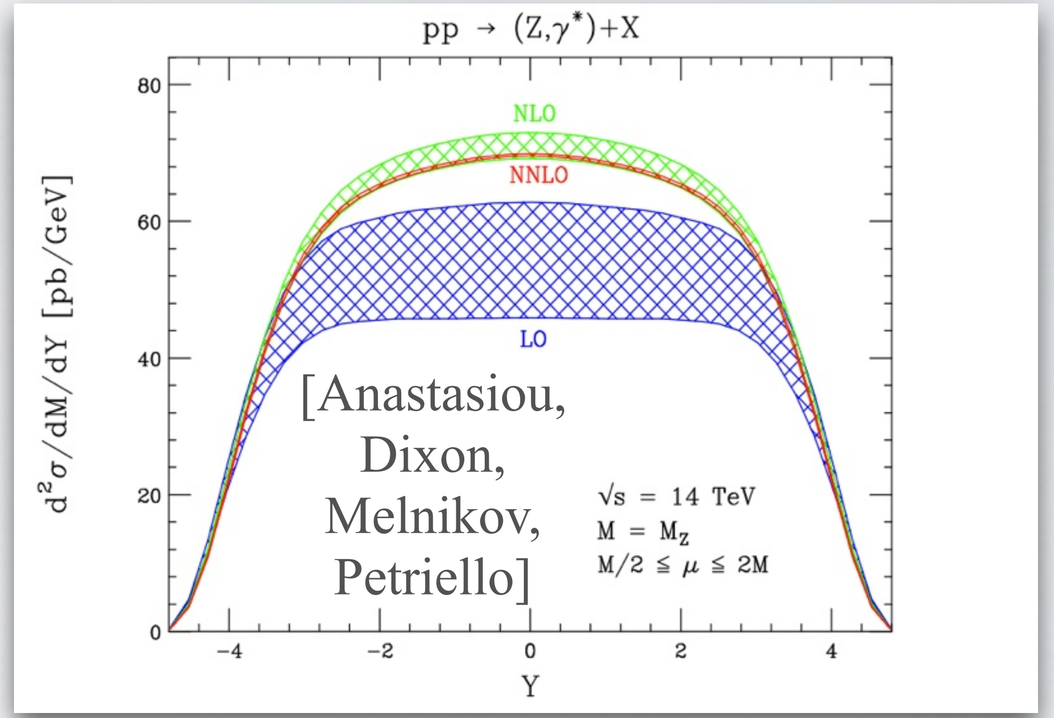
Amazing progress from the experiments

State of QCD perturbation theory

$$\sigma = \int dx_1 dx_2 \mathrm{pdf}(x_1)\mathrm{pdf}(x_2)\boxed{\hat{\sigma}(x_1 x_2)}$$

- Impressive repository of calculations available at NLO (MCFM)
- Tree level and NLO calculations almost completely automated
- At NNLO only a few processes are done: Higgs production, Drell-Yan, top pair production,…
- Calculations are specific for each process
- More general methods are slowly being pushed to NNLO (unitarity, …)
- First proof of principle calculations are being done

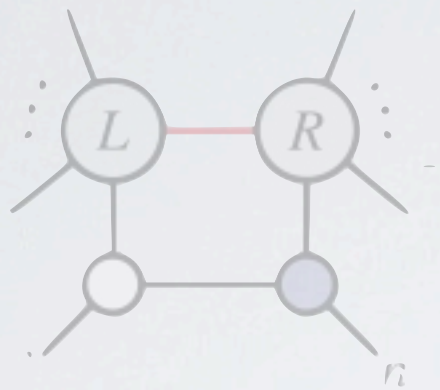
State of QCD perturbation theory

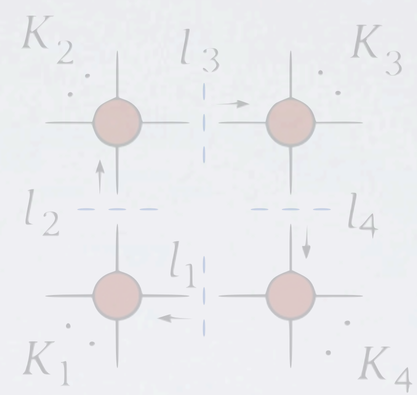


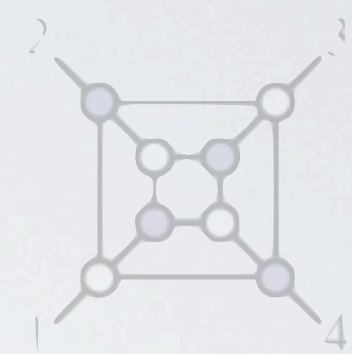
- Perturbation theory can be tedious
- But it is necessary
- We have seen significant corrections from the classical level to NLO in many processes
- And even remarkable corrections from NLO to NNLO for example in Higgs production through gluon fusions

Loop calculations are important for phenomenological predictions

State of QCD perturbation theory







- Loop amplitudes are a probe into the inner workings of QFT
- Shed light on the structure of gauge theories at higher loop orders
- Find and test conjectures about all-loop structure of gauge theories (planar N=4 Super Yang-Mills)

Formal interest in loop calculations

Loop amplitudes

- Practical and formal interest in loop amplitudes
- Loop computations are notoriously difficult
    - Explosive growth of the number of Feynman diagrams
    - The integrals are in general UV & IR divergent
    - Computing the integrals is in general extremely difficult
- I will mostly talk about the actual integrals in this talk

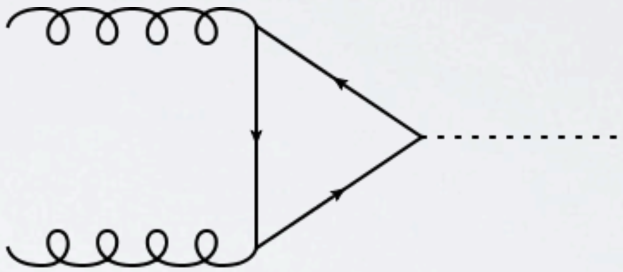
Loop amplitudes

- Why are loop integrals difficult to compute?
- Loop integrals cannot be described by elementary functions
- A whole zoo of transcendental functions has been observed in calculations of loop integrals
  - Classical polylogarithms
  - Harmonic polylogarithms (HPLs)
  - Cyclotomic harmonic polylogarithms
  - 2d harmonic polylogarithms
  - These are all special cases of the **multiple polylogarithms**
  - Elliptic functions (not in this talk)
- What are the properties of these functions?
- **How can we perform integrals involving these functions?**

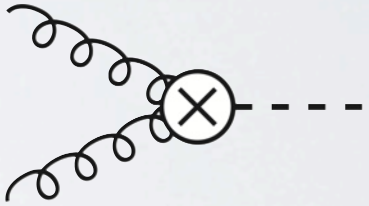
- Why do we compute? ✔
- What do we want to compute?
- How do we compute it?
- What do we find?

The gluon fusion cross section

- The dominant Higgs production mode at the LHC is gluon fusion
  - Loop-induced process



- The Higgs boson is light compared to the top quark
- The top loop can be integrated out → effective theory



- The tree-level coupling of the gluons to the Higgs is described by a dimension five operator

$$\mathcal{L} = \mathcal{L}_{\mathrm{QCD}} - \frac{1}{4v} C_1 H G^a_{\mu\nu} G^{\mu\nu}_a$$

The gluon fusion cross section

- Operators with higher dimension can be included in the computation
- This leads to a systematic expansion of the gluon fusion cross section in the top mass
- Sub-leading corrections in the top-mass are known at NNLO

[Harlander, Ozeren; Pak, Rogal, Steinhauser; Ball, Del Duca, Marzani, Forte, Vicini; Harlander, Mantler, Marzani, Ozeren]

- In the following I will only talk about the leading term in the effective theory

The gluon fusion cross section

- The gluon fusion cross-section in perturbation theory is

$$\sigma\left(pp \to H + X\right) = \tau \sum_{ij} \int_{\tau}^{1} dz \mathcal{L}_{ij}(z) \hat{\sigma}_{ij}\left(\frac{\tau}{z}\right)$$

- We compute the inclusive partonic cross section

- The partonic cross section is a function of

$$z = \frac{m_h^2}{\hat{s}} \qquad \tau = \frac{m_h^2}{E_{cm}^2}$$

- In perturbation theory the partonic cross section can be expanded

$$\hat{\sigma}(z) = \hat{\sigma}^{\mathrm{LO}}(z) + \alpha_s \hat{\sigma}^{\mathrm{NLO}}(z) + \alpha_s^2 \hat{\sigma}^{\mathrm{NNLO}}(z) + \alpha_s^3 \hat{\sigma}^{\mathrm{N3LO}}(z) + \ldots$$

The gluon fusion cross section

- The lower orders of the gluon fusion cross section have been computed
  - NLO (full theory)

    [Dawson; Djouadi, Spira, Zerwas]
  - NNLO (effective theory and sub-leading top-mass corrections)

    [Harlander, Kilgore; Anastasiou, Melnikov; Ravindran, Smith, van Neerven]

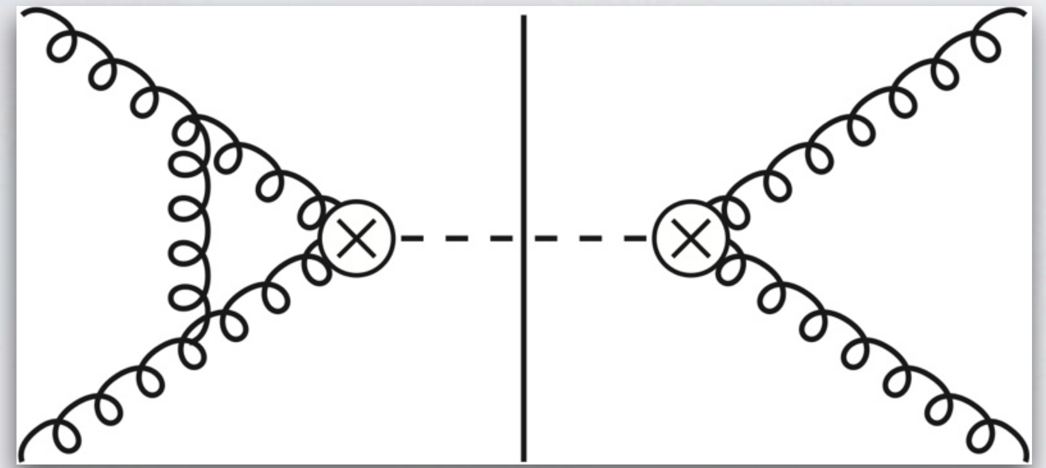
fixed order only

	σ [8 TeV]	$\delta\sigma$ [%]
LO	9.6 pb	~ 25%
NLO	16.7 pb	~ 20%
NNLO	19.6 pb	~ 7 - 9%
N3LO	???	~ 4 - 8%

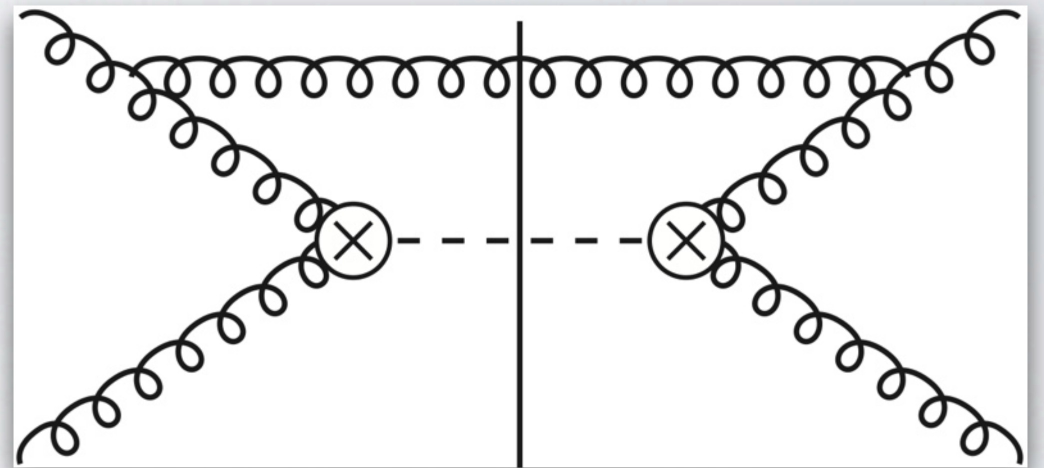
- We want to push the calculation one order higher
- Uncharted territory in perturbation theory
- Many conceptual and practical challenges

The gluon fusion cross section

- Diagrammatic contributions at NLO



virtual correction

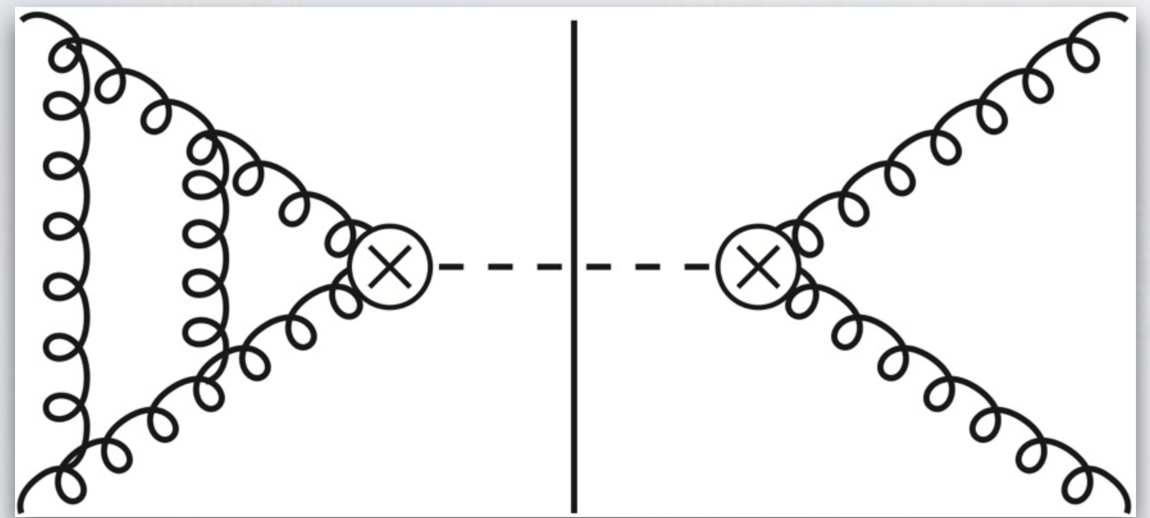


real emission

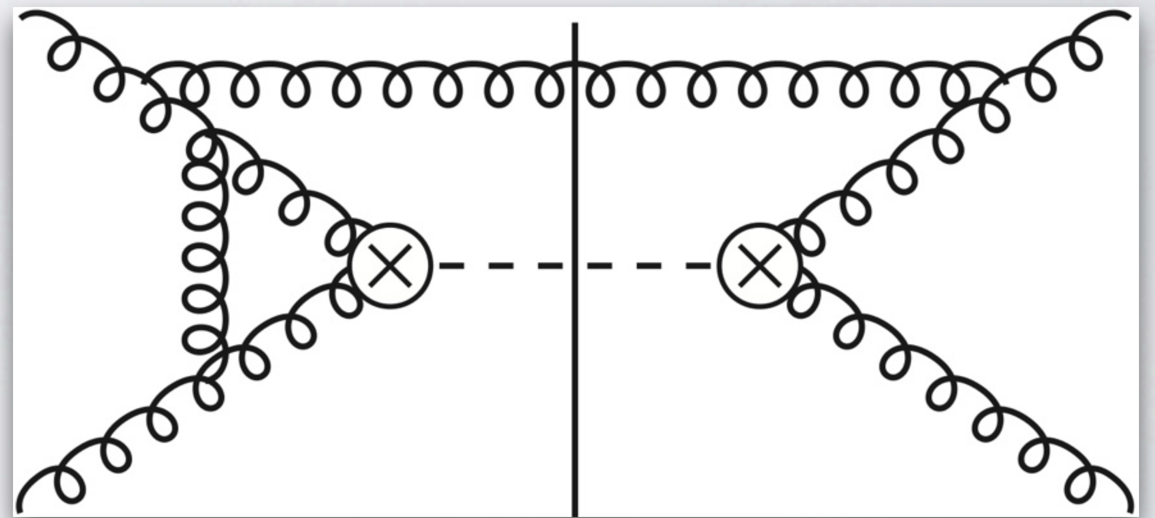
- Purely diagrammatic contributions are divergent
- Need two more pieces in the calculation
  - UV renormalization to cancel UV divergences
  - PDF counter terms to cancel initial state collinear (IR) divergences

The gluon fusion cross section

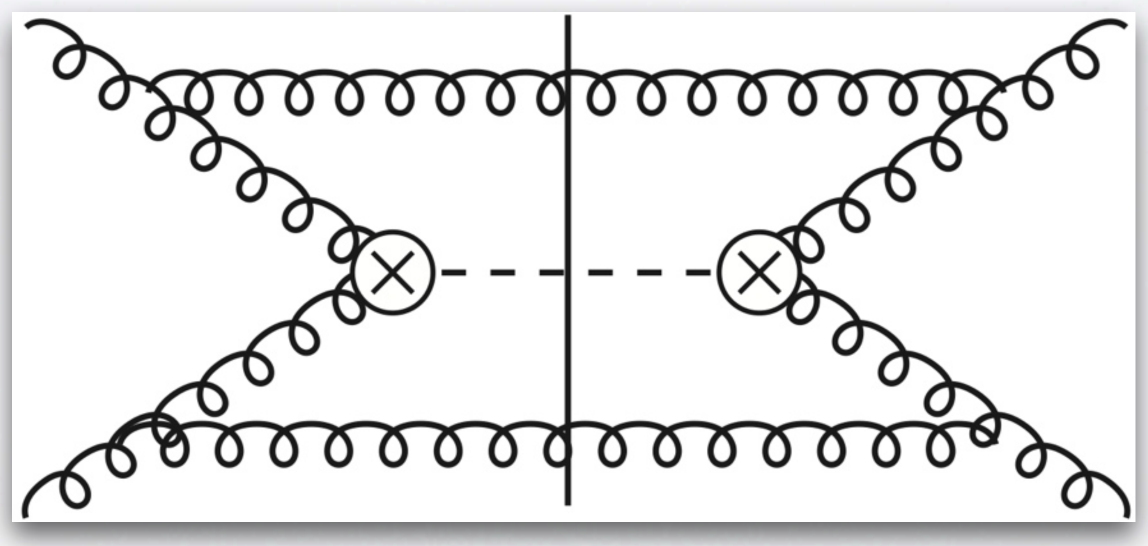
- Diagrammatic contributions at NNLO



double virtual



real virtual



double real

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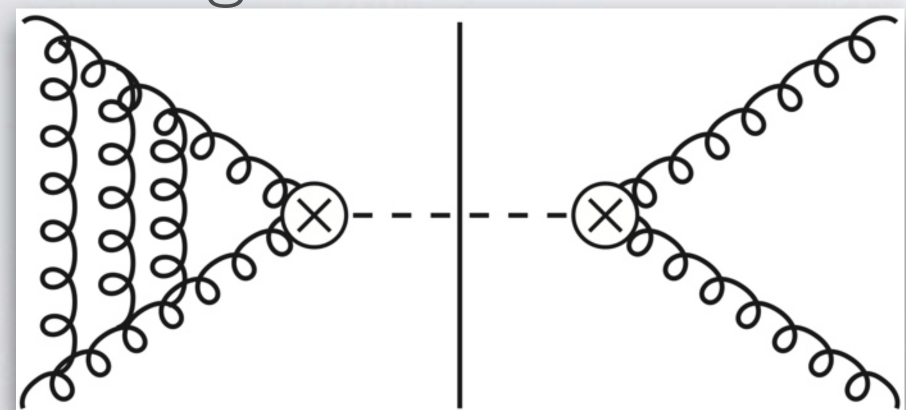
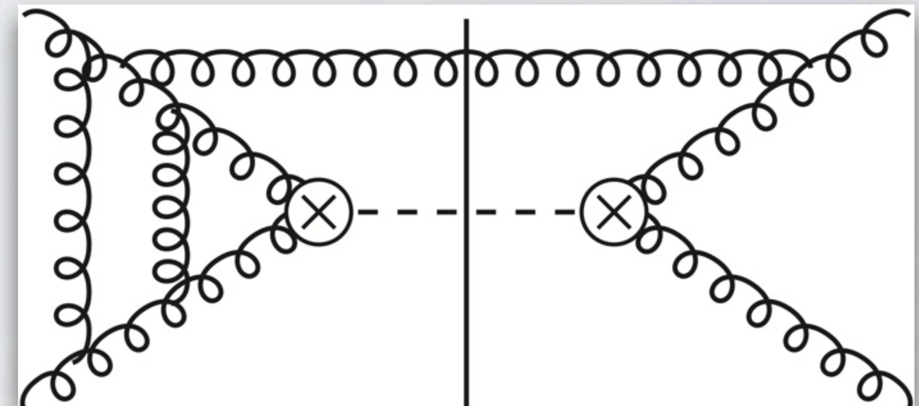
The gluon fusion cross section

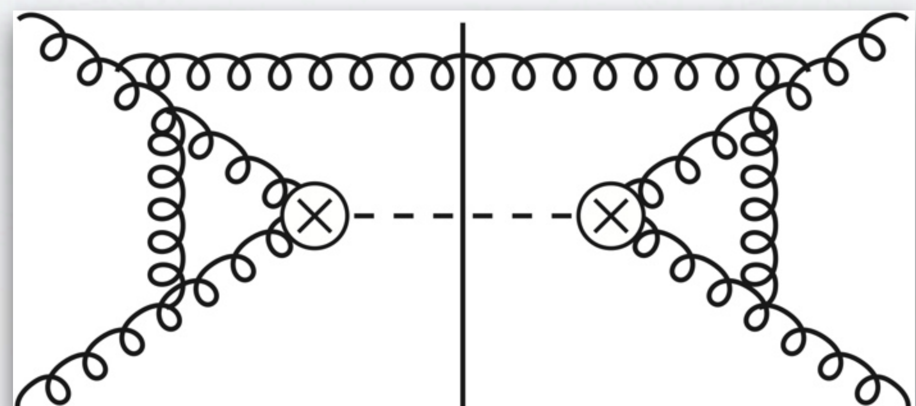
- Diagrammatic contributions at NNNLO



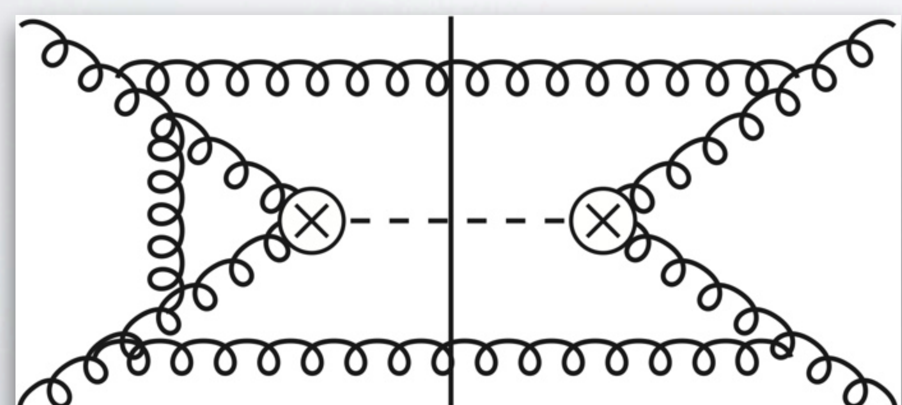
triple virtual



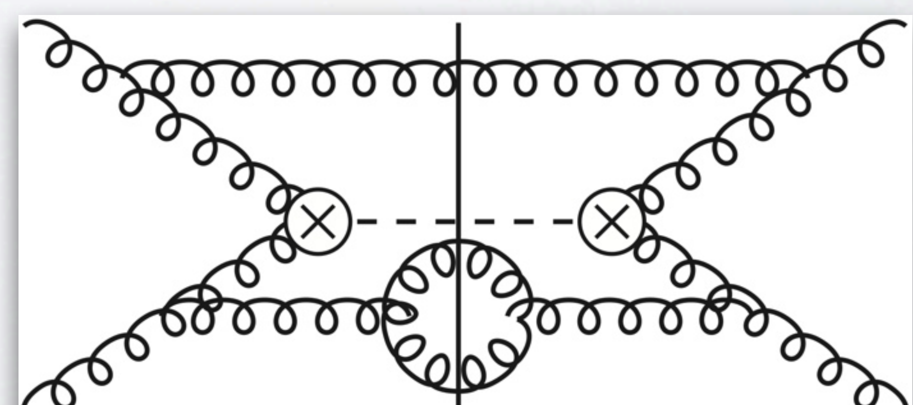
double virtual real



real virtual squared



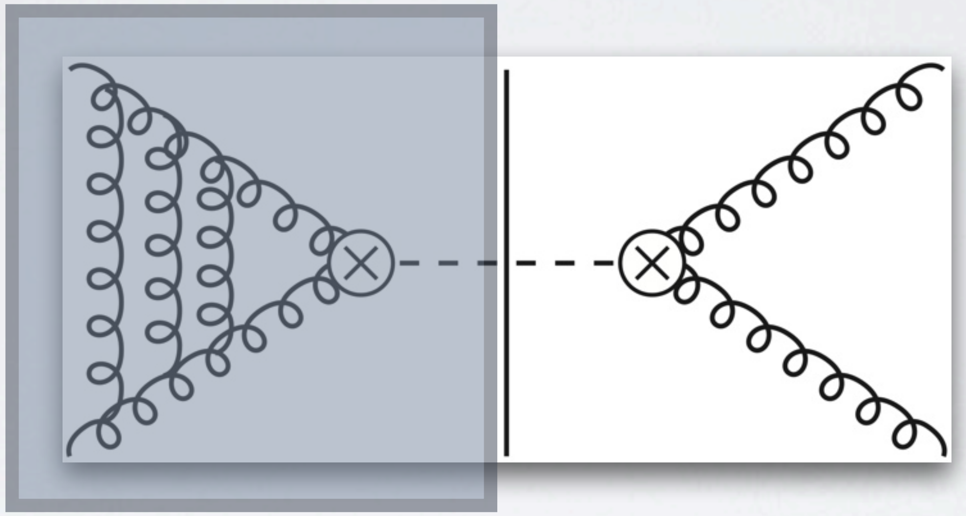
double real virtual



triple real

The triple virtual

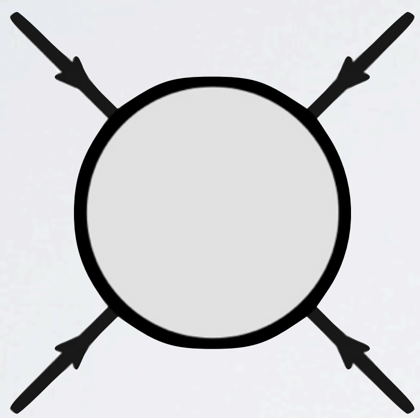
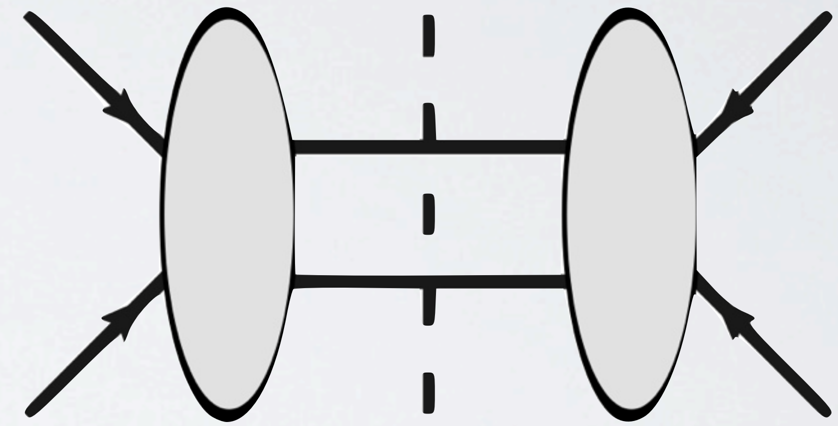
- The triple virtual is directly related to the three loop QCD form factor



- The QCD form factor is well known
    - at one loop
    - at two loops [Gonsalves; Kramer, Lampe; Gehrmann, Huber, Maitre]
    - at three loops [Baikov, Chetyrkin, Smirnov, Smirnov, Steinhauser; Gehrmann, Glover, Huber, Ikizlerli, Studerus]
- The pure loop contributions are not a problem in the calculation

Unitarity

- Optical theorem:

Im  $= \int d\Phi$ 

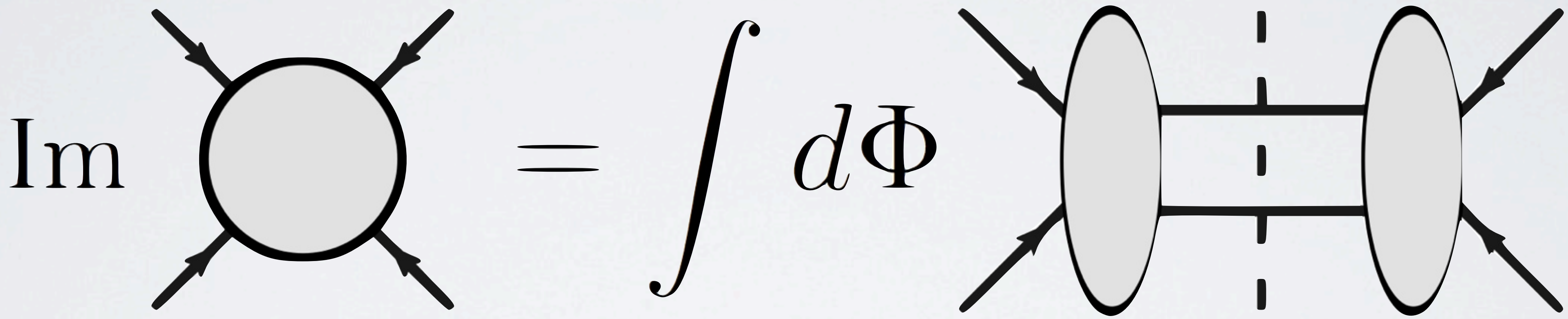
  - Discontinuities of loop integrals are phase space integrals

- Discontinuities of loop integrals are given by Cutkosky's rule:

$$\frac{1}{p^2 - m^2 + i\epsilon} \to \delta^+(p^2 - m^2) = \delta(p^2 - m^2)\theta(p^0)$$

Reverse unitarity

- Optical theorem:



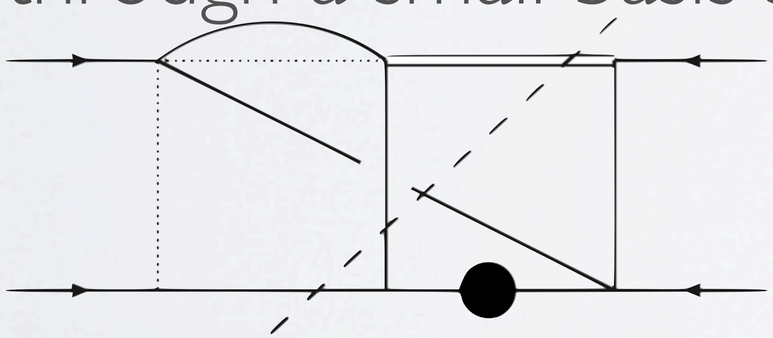
- The optical theorem can be read 'backwards'
- This way, phase space integrals can be expressed as unitarity cuts of loop integrals [Anastasiou, Melnikov; Anastasiou, Dixon, Melnikov, Petriello]
- We can compute loop integrals with cuts instead of phase space integrals
- This makes the rich technology developed for loop integrals available

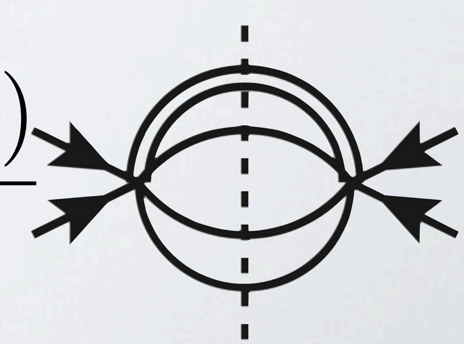
IBPs and master integrals

- Loop integrals are in general not independent but related by Integration-by-parts identities (IBPs)

- The IBPs form a system of equations for a given class of loop integrals

- The system can be solved algorithmically expressing all integrals through a small basis set of integrals (master integrals)



- $= -\dfrac{(\epsilon-1)(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)(6\epsilon-5)(6\epsilon-1)}{\epsilon^4(\epsilon+1)(2\epsilon-3)}$ 

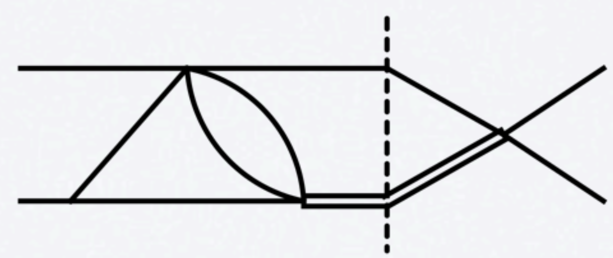
IBPs and differential equations

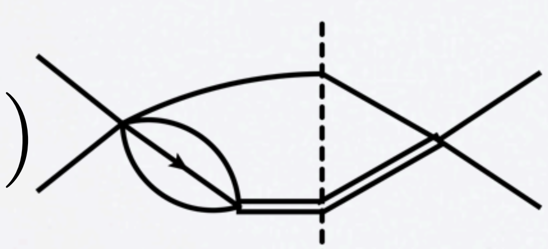
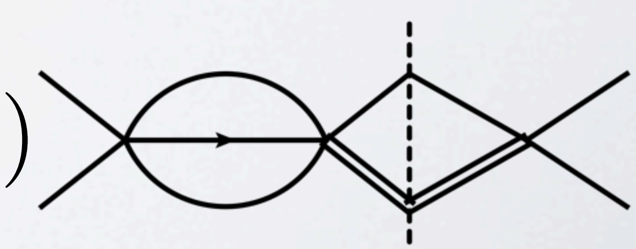
- Having access to IBP technology allows us to derive differential equations for master integrals

- The derivative of a master integral w.r.t. kinematic invariants can be expressed as a linear combination of master integrals

- Leads to a coupled system of linear differential equations for the master integrals

$$\bar{z} = 1 - z = \frac{s - m_h^2}{s}$$

$$\left[\partial_{\bar{z}} - 3\epsilon\ \mathrm{dlog}(1-\bar{z})\right]$$ 

- $$= \epsilon\ \mathrm{dlog}(1-\bar{z})$$  $$-3\epsilon\ \mathrm{dlog}(1-\bar{z})$$ 

Differential equations and boundaries

- Integrating the differential equations for the master integrals yields general solutions
- These general solutions need to be fixed using boundary conditions
- Natural boundary condition for the problem at

$$\bar{z} = 0 \Longleftrightarrow \hat{s} = m_h^2$$

- This corresponds to the soft or threshold limit of the process

The threshold expansion

- It is possible to systematically expand the cross section at threshold
- This yields
  - boundary conditions for the differential equation
  - the soft-virtual approximation for the cross-section
- Around threshold the cross section can be approximated by a power series

$$\hat{\sigma} = \hat{\sigma}_{-1} + \hat{\sigma}_0 + \bar{z}\hat{\sigma}_1 + \mathcal{O}(\bar{z})^2$$

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The soft-virtual approximation

- We computed the soft-virtual term

$$\hat{\sigma} = \boxed{\hat{\sigma}_{-1}} + \hat{\sigma}_0 + \bar{z}\hat{\sigma}_1 + \mathcal{O}(\bar{z})^2$$

- It receives contributions from the threshold divergence at $\bar{z} \sim 0$

- The poles in epsilon cancel with the poles from the virtual contributions to leave behind delta functions and plus distributions

$$\bar{z}^{-1+n\epsilon} = \frac{\delta(\bar{z})}{n\epsilon} + \left[\frac{1}{\bar{z}}\right]_+ + n\epsilon\left[\frac{\log(\bar{z})}{\bar{z}}\right]_+ + \mathcal{O}(\epsilon)^2$$

- The soft-virtual term includes:
  - The full three-loop corrections to gluon fusion
  - Contributions from the real emission of soft gluons at up to two loops
  - Only gluon initiated channels contribute

The soft-virtual approximation

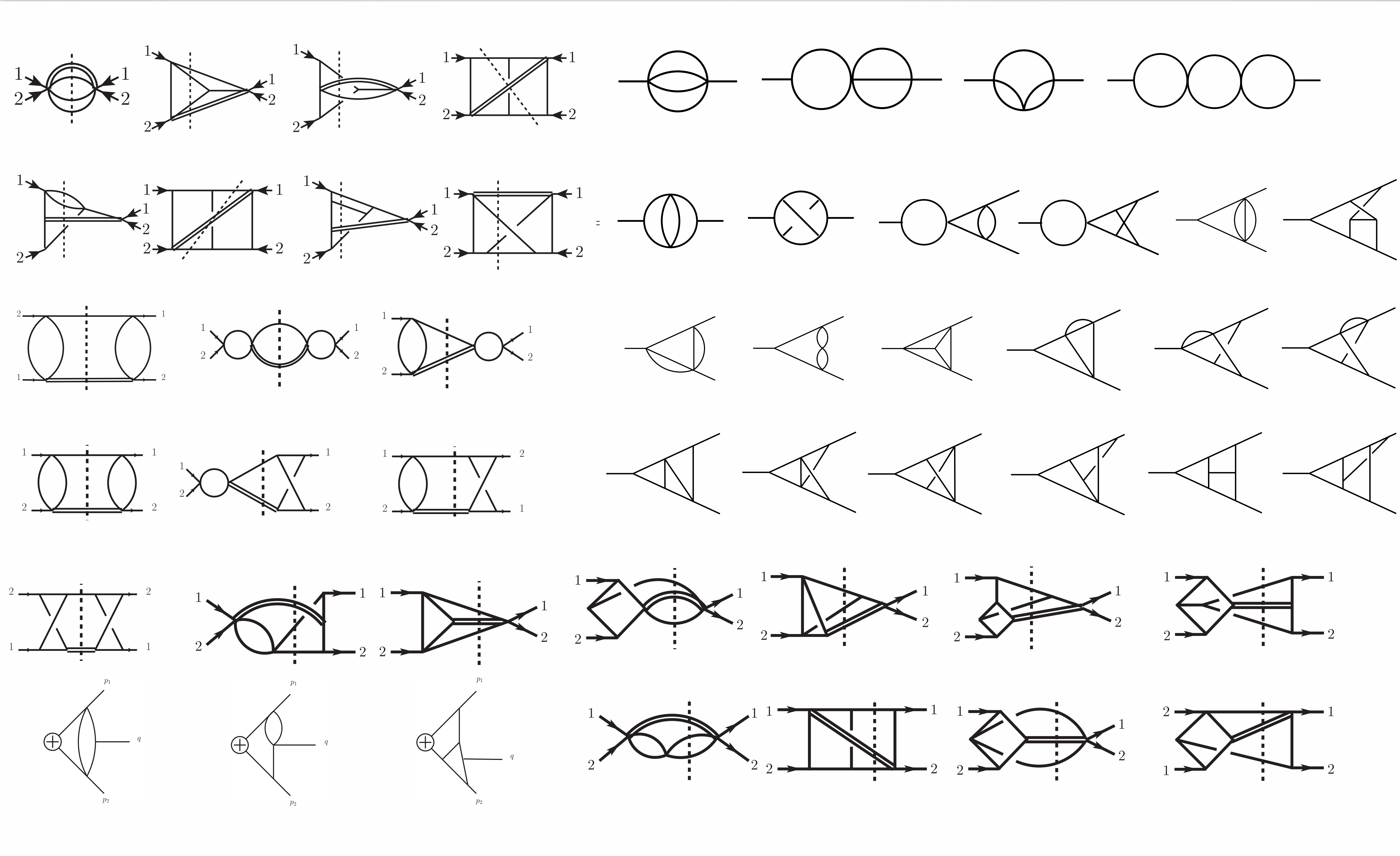
- Expansion by regions provides a consistent way to compute the soft contribution to the cross section:
  - Expand the integrals in soft momenta
  - The momenta of final state partons are soft
  - Loop momenta are either soft or hard compared to the final state parton momenta
  - The expanded objects can be interpreted as Feynman integrals themselves
- Expansion reduces the complexity of the calculation
  - Less master integrals
  - Simpler master integrals

The soft-virtual approximation

- All required integrals can be computed analytically
  - 22 three-loop integrals [Baikov, Chetyrkin, Smirnov, Smirnov, Steinhauser; Gehrmann, Glover, Huber, Ikizlerli, Studerus]
  - 3 double-virtual real integrals [Duhr, Gehrmann; Li, Zhu]
  - 7 real-virtual squared integrals [Anastasiou, Duhr, FD, Herzog, Mistlberger; Kilgore]
  - 10 double-real virtual integrals [Anastasiou, Duhr, FD, Herzog, Mistlberger; Li, von Manteufel, Schabinger, Zhu]
  - 8 triple real integrals [Anastasiou, Duhr, FD, Mistlberger]

- Additionally
  - three-loop splitting functions [Moch, Vogt, Vermaseren]
  - three-loop beta functions [Tarasov, Vladimirov, Zharkov; Larin, Vermaseren]
  - three-loop Wilson coefficient [Chetyrkin, Kniehl, Steinhauser; Schroder, Steinhauser; Chetyrkin, Kuhn, Sturm]

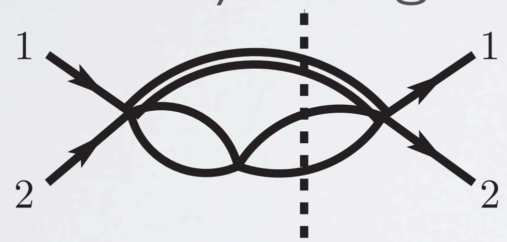
The master integrals



- Why do we compute? ✔
- What do we want to compute? ✔
- How do we compute it?
- What do we find?

The integrals

- We want to compute all the integrals analytically
- Every integral is individually divergent and gives rise to up to six poles in dimensional regularization
- Many integrals are trivial to compute:



$$= \frac{\Gamma(4-4\epsilon)\Gamma(2-3\epsilon)}{\epsilon(1-2\epsilon)^2\Gamma(4-6\epsilon)\Gamma(1-\epsilon)}$$

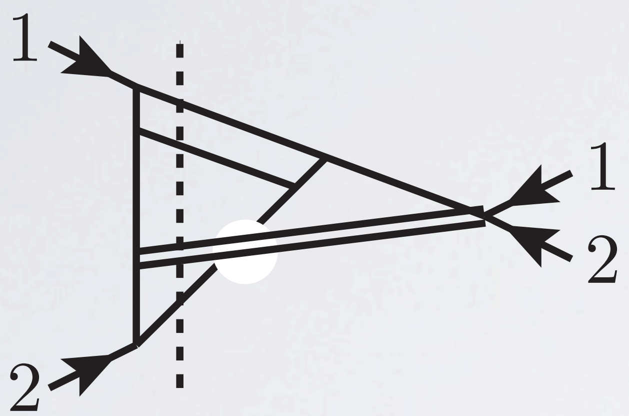
$$= \frac{1}{\epsilon} + \frac{14}{3} + (24-6\zeta_2)\epsilon + \left(-28\zeta_2 - 42\zeta_3 + \frac{400}{3}\right)\epsilon^2 + (-144\zeta_2 - 196\zeta_3$$

$$-195\zeta_4 + \frac{2320}{3}\Big)\,\epsilon^3 + (252\zeta_3\zeta_2 - 800\zeta_2 - 1008\zeta_3 - 910\zeta_4 - 1302\zeta_5 + 4576)\,\epsilon^4$$

- $$+\left(882\zeta_3^2 + 1176\zeta_2\zeta_3 - 5600\zeta_3 - 4640\zeta_2 - 4680\zeta_4 - 6076\zeta_5 - \frac{9219}{2}\zeta_6 + 81920\right)\epsilon^5 + \mathcal{O}(\epsilon)^6$$

The integrals

- Other integrals not so much



$$= \frac{\Gamma(12-6\epsilon)\Gamma(3-3\epsilon)\Gamma(1-\epsilon)}{\Gamma(5-6\epsilon)\Gamma(2-\epsilon)^4}\Big[\mathcal{I}_{9,1}(\epsilon)+\mathcal{I}_{9,2}(\epsilon)\Big]$$

$$\begin{aligned}
\mathcal{I}_{9,1}(\epsilon) = -\int_0^\infty dt_1\,dt_2 \int_0^1 dx_1\,dx_2\,dx_3\, t_1^{2-4\epsilon}\,(1+t_1)^{\epsilon-1}\,t_2^{1-2\epsilon} \\
\times x_1^{-\epsilon}\,(1-x_1)^{2-4\epsilon}\,x_2^{1-3\epsilon}\,(1-x_2)^{-\epsilon}\,x_3^{-\epsilon}\,(1+t_2x_3)^{1-3\epsilon}\,(1+t_2x_2x_3)^{\epsilon} \\
\times \left(t_1t_2^2x_1x_2x_3+t_2^2x_2x_3+t_1t_2x_1x_2+t_1t_2x_3+t_2x_2x_3+t_2+t_1+1\right)^{3\epsilon-3}\,, \\
\mathcal{I}_{9,2}(\epsilon) = \int_0^\infty dt_1\,dt_2 \int_0^1 dx_1\,dx_2\,dx_3\, t_1^{2-4\epsilon}\,(1+t_1)^{\epsilon-1}\,t_2^{1-2\epsilon} \\
\times x_1^{1-\epsilon}\,(1-x_1)^{2-4\epsilon}\,x_2^{1-3\epsilon}\,(1-x_2)^{-\epsilon}\,x_3^{-\epsilon}\,(1+t_2x_3)^{1-3\epsilon}\,(1+t_2x_2x_3)^{\epsilon} \\
\times \left(t_1t_2^2x_1x_2x_3+t_2^2x_1x_2x_3+t_2x_1+t_1t_2x_1x_2+t_1t_2x_3+t_2x_1x_2x_3+t_1+x_1\right)^{3\epsilon-3}\,,
\end{aligned}$$

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The integrals

- Why are some of these iterated integrals so complicated?

- In the previous example we have 5 integrations
  - 1st integration $\int \frac{dx_1}{x_1} \to \log$
  - 2nd integration $\int \frac{dx_2}{x_2} \log\left(f(x_2)\right) \to \mathrm{Li}_2$
  - 3rd integration $\int \frac{dx_3}{x_3} \mathrm{Li}_2\left(f(x_3)\right) \to \mathrm{Li}_3$
- With each integration step we obtain more and more complicated functions

$$\mathrm{Li}_n(z) = \int_0^z \frac{dt}{t} \mathrm{Li}_{n-1}(t)$$

- Depending on the problem, we find even more complex functions than the classical polylogarithms

Multiple polylogarithms

- Large classes of loop integrals can be expressed in terms of multiple polylogarithms

$$G\left(a_1, \ldots, a_n; z\right) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t) \;\Big|\; \mathrm{Li}_n(z) = \int_0^z \frac{dt}{t} \mathrm{Li}_{n-1}(t)$$

- The classical polylogarithms, HPLs, 2dHPLs etc are special cases of the multiple polylogarithms

- The classical polylogarithms satisfy various complicated functional identities

$$-\mathrm{Li}_2(z) - \log(z)\log(1-z) = \mathrm{Li}_2(1-z) - \frac{\pi^2}{6}$$

- For the multiple polylogarithms these identities are in general not known

Multiple polylogarithms

- Not knowing the functional identities is a problem
- Even if the physics of a result is very simple, the analytical expression might be very complicated
  - The simplicity of the answer might be hidden behind the various functional equations
  - Famous example:
    - The two-loop hexagon remainder function in N=4 SYM as computed by Del Duca, Duhr and Smirnov is a 17 page expression
    - After Goncharov, Spradlin, Vergu and Volovich simplified it using functional identities it can be written in 4 lines

Multiple polylogarithms

- Not knowing the functional identities is a problem
- Too complicated results are not just a formal or aesthetic problem
- Without using functional identities there might be huge cancellation between divergent sub-pieces of the result even though the complete result is finite
- Too complicated results are not useable for phenomenology because numerical implementations are not feasible
- Need functional identities to express result in a simple basis

Multiple polylogarithms

- Not knowing the functional identities is a problem
- The integrand might not be in the right form to perform the integration
- Result can only be obtained if functional identities between polylogarithms are known

$$G\left(a_1, \ldots, a_n; z\right) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)$$

Number theory

- Multiple polylogarithms are a very active field of research in pure mathematics
- Mathematicians have discovered algebraic structures that underly the polylogarithms
- When we usually think of functional identities we think of complicated functional equations that are obtained by performing intricate variable transformations of the integral representations
- $$-\mathrm{Li}_2(z) - \log(z)\log(1-z) = \mathrm{Li}_2(1-z) - \frac{\pi^2}{6}$$

Number theory

- Mathematicians have conjectured that all functional equations between polylogarithms follow from a simple algebraic structure
- All functional equations between polylogarithms can be obtained from pure combinatorics
- One need not even know the integral to obtain the functional identities
- The algebraic structure that governs the polylogarithms is called a **Hopf algebra**

Hopf algebras

- What is a **Hopf algebra**?

- It is an **algebra**: A vector space with an operation that allows us to combine two elements into one (multiplication)

- It is also a **coalgebra**: A vector space with an operation that allows us to break an element into two elements (comultiplication)

- Disclaimer: The following explanation is very handwaving and omits many mathematical details

Hopf algebras

- The algebra part of the Hopf algebra manifests itself as the **shuffle algebra** of the polylogarithms

- Shuffle product: Takes two words and intersperses them in all possible ways while keeping the ordering of the letters of each word among themselves

$$ab \sqcup\!\sqcup\, cd = abcd + acbd + acdb + cabd + cadb + adab$$

- Analogy: Riffle shuffling two stacks of cards.

$$\log(x)\log(1-x)$$
$$= -G(0,1,x) - G(1,0,x)$$

-

Hopf algebras

- The comultiplication of the Hopf algebra for polylogarithms is called the **coproduct** [Duhr]

- It splits a word in all possible ordered ways

$$\Delta(abcd) = abcd \otimes 1 + abc \otimes d + ab \otimes cd + a \otimes bcd + 1 \otimes abcd$$

- We can iterate this splitting until we have broken the word into tensor products of single letters

Functional equations

- The coproduct can be applied to polylogarithms [Duhr]
- The word is here the list of indices $\{a_n\}$ of a polylogarithm

$$G(a_1, \ldots, a_n; z)$$

- Examples:

$$\Delta(\log x) = 1 \otimes \log x + \log x \otimes 1$$

$$\Delta(\mathrm{Li}_2(x)) = 1 \otimes \mathrm{Li}_2(x) - \log(1-x) \otimes \log(x) + 1 \otimes \mathrm{Li}_2(x)$$

-

Functional equations

- The coproduct can be used to derive functional equations for polylogarithms
- The coproduct is applied to the polylogarithm to split it into simpler pieces
- The functional identities for these simpler pieces might be known
- If not, the coproduct is repeatedly applied until only ordinary logarithms are left

Example

- Assume you want to calculate: $$\int_0^1 dx \frac{\mathrm{Li}_2\left(\frac{ax}{(1-x)}\right)}{x(1-x)}$$

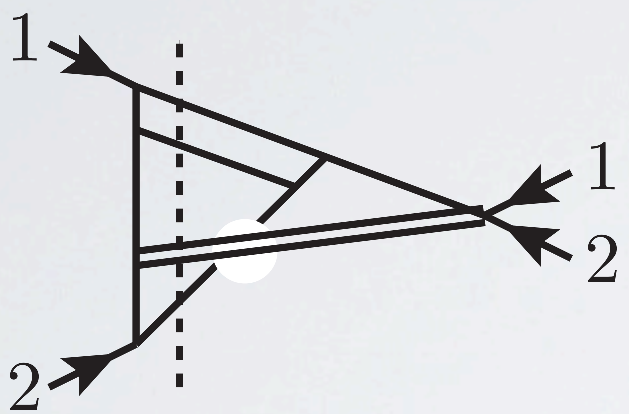
- Using the coproduct it is possible to derive the following functional identity:

$$\mathrm{Li}_2\left(\frac{ax}{(1-x)}\right) = G\left(0,1;x\right) - G\left(0,\frac{1}{1+a};x\right) - G\left(1,1;x\right) + G\left(1,\frac{1}{1+a};x\right)$$

- Now all the integrations are trivial:

- $$G\left(a_1,\ldots,a_n;z\right) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_n;t)$$

The integrals



$$= \frac{\Gamma(12-6\epsilon)\Gamma(3-3\epsilon)\Gamma(1-\epsilon)}{\Gamma(5-6\epsilon)\Gamma(2-\epsilon)^4}\Big[\mathcal{I}_{9,1}(\epsilon)+\mathcal{I}_{9,2}(\epsilon)\Big]$$

$$\mathcal{I}_{9,1}(\epsilon) = -\int_0^\infty dt_1\, dt_2 \int_0^1 dx_1\, dx_2\, dx_3\, t_1^{2-4\epsilon}\,(1+t_1)^{\epsilon-1}\, t_2^{1-2\epsilon}$$
$$\times\, x_1^{-\epsilon}\,(1-x_1)^{2-4\epsilon}\, x_2^{1-3\epsilon}\,(1-x_2)^{-\epsilon}\, x_3^{-\epsilon}\,(1+t_2x_3)^{1-3\epsilon}\,(1+t_2x_2x_3)^{\epsilon}$$
$$\times\,\Big(t_1t_2^2x_1x_2x_3+t_2^2x_2x_3+t_1t_2x_1x_2+t_1t_2x_3+t_2x_2x_3+t_2+t_1+1\Big)^{3\epsilon-3}\,,$$

$$\mathcal{I}_{9,2}(\epsilon) = \int_0^\infty dt_1\, dt_2 \int_0^1 dx_1\, dx_2\, dx_3\, t_1^{2-4\epsilon}\,(1+t_1)^{\epsilon-1}\, t_2^{1-2\epsilon}$$
$$\times\, x_1^{1-\epsilon}\,(1-x_1)^{2-4\epsilon}\, x_2^{1-3\epsilon}\,(1-x_2)^{-\epsilon}\, x_3^{-\epsilon}\,(1+t_2x_3)^{1-3\epsilon}\,(1+t_2x_2x_3)^{\epsilon}$$

- $$\times\,\Big(t_1t_2^2x_1x_2x_3+t_2^2x_1x_2x_3+t_2x_1+t_1t_2x_1x_2+t_1t_2x_3+t_2x_1x_2x_3+t_1+x_1\Big)^{3\epsilon-3}\,,$$

Number theory

- Number theory helps us here
- The integral can be done one variable at a time
- We use the coproduct to derive the needed functional identities at each step
- Integrate over one variable at a time using the basic definition of the multiple polylogarithms

$$G\left(a_1, \ldots, a_n; z\right) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)$$

- Number theory gives us a way to solve the integrals algorithmically

[Brown]

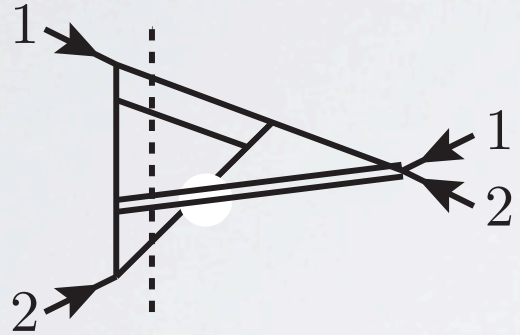
Number theory

- The previous integral can be computed one step at a time
- In the process one finds functional identities like:
- Such identities can not be found in the literature
- Nobody wants to derive them using integral transformations
- Number theory and the coproduct give us a simple

$$
\begin{aligned}
&G\left(\frac{tux+ux-u+1}{x(tu+1)}, \frac{tux+ux-u+1}{x(tu+1)}, -\frac{1}{tx}, 1, 1\right) = \\
&-G\left(0,1,1,-\frac{1}{t},x\right) - G\left(0,1,-\frac{1}{tu},1,x\right) + G\left(0,1,-\frac{1}{tu},-\frac{1}{t},x\right) + G\left(0,1,-\frac{1}{tu},-\frac{1-u}{tu+u},x\right) - G\left(0,-\frac{1}{tu},1,1,x\right) + \\
&G\left(0,-\frac{1}{tu},1,-\frac{1}{t},x\right) + G\left(0,-\frac{1}{tu},1,-\frac{1-u}{tu+u},x\right) + G\left(0,-\frac{1}{tu},-\frac{1}{tu},1,x\right) - G\left(0,-\frac{1}{tu},-\frac{1}{tu},-\frac{1}{t},x\right) - \\
&G\left(0,-\frac{1}{tu},-\frac{1}{tu},-\frac{1-u}{tu+u},x\right) + G\left(0,-\frac{1}{tu},-\frac{1-u}{tu+u},1,x\right) - G\left(0,-\frac{1}{tu},-\frac{1-u}{tu+u},-\frac{1-u}{tu+u},x\right) - G\left(1,0,1,-\frac{1}{t},x\right) - \\
&G\left(1,0,-\frac{1}{tu},1,x\right) + G\left(1,0,-\frac{1}{tu},-\frac{1}{t},x\right) + G\left(1,0,-\frac{1}{tu},-\frac{1-u}{tu+u},x\right) - G\left(1,1,0,-\frac{1}{t},x\right) + G\left(1,1,-\frac{1}{t},-\frac{1}{t},x\right) + \\
&G\left(1,-\frac{1}{t},0,1,x\right) - G\left(1,-\frac{1}{t},0,-\frac{1-u}{tu+u},x\right) - G\left(1,-\frac{1}{t},1,1,x\right) + G\left(1,-\frac{1}{t},1,-\frac{1}{t},x\right) + G\left(1,-\frac{1}{t},1,-\frac{1-u}{tu+u},x\right) + \\
&G\left(1,-\frac{1}{t},-\frac{1}{tu},1,x\right) - G\left(1,-\frac{1}{t},-\frac{1}{tu},-\frac{1}{t},x\right) - G\left(1,-\frac{1}{t},-\frac{1}{tu},-\frac{1-u}{tu+u},x\right) - G\left(1,-\frac{1}{tu},0,1,x\right) + G\left(1,-\frac{1}{tu},0,-\frac{1}{t},x\right) + \\
&G\left(1,-\frac{1}{tu},0,-\frac{1-u}{tu+u},x\right) + G\left(1,-\frac{1}{tu},1,1,x\right) - G\left(1,-\frac{1}{tu},1,-\frac{1-u}{tu+u},x\right) - G\left(1,-\frac{1}{tu},-\frac{1}{t},-\frac{1}{t},x\right) + G\left(-\frac{1}{t},0,1,1,x\right) - \\
&G\left(-\frac{1}{t},0,1,-\frac{1-u}{tu+u},x\right) - G\left(-\frac{1}{t},0,-\frac{1-u}{tu+u},1,x\right) + G\left(-\frac{1}{t},0,-\frac{1-u}{tu+u},-\frac{1-u}{tu+u},x\right) - G\left(-\frac{1}{t},1,1,1,x\right) + G\left(-\frac{1}{t},1,1,-\frac{1}{t},x\right) + \\
&G\left(-\frac{1}{t},1,1,-\frac{1-u}{tu+u},x\right) + G\left(-\frac{1}{t},1,-\frac{1}{tu},1,x\right) - G\left(-\frac{1}{t},1,-\frac{1}{tu},-\frac{1}{t},x\right) - G\left(-\frac{1}{t},1,-\frac{1}{tu},-\frac{1-u}{tu+u},x\right) + G\left(-\frac{1}{t},1,-\frac{1-u}{tu+u},1,x\right) - \\
&G\left(-\frac{1}{t},1,-\frac{1-u}{tu+u},-\frac{1-u}{tu+u},x\right) + G\left(-\frac{1}{t},-\frac{1}{tu},1,1,x\right) - G\left(-\frac{1}{t},-\frac{1}{tu},1,-\frac{1}{t},x\right) - G\left(-\frac{1}{t},-\frac{1}{tu},1,-\frac{1-u}{tu+u},x\right) - \\
&G\left(-\frac{1}{t},-\frac{1}{tu},-\frac{1}{tu},1,x\right) + G\left(-\frac{1}{t},-\frac{1}{tu},-\frac{1}{tu},-\frac{1}{t},x\right) + G\left(-\frac{1}{t},-\frac{1}{tu},-\frac{1}{tu},-\frac{1-u}{tu+u},x\right) - G\left(-\frac{1}{t},-\frac{1}{tu},-\frac{1-u}{tu+u},1,x\right) + \\
&G\left(-\frac{1}{t},-\frac{1}{tu},-\frac{1-u}{tu+u},-\frac{1-u}{tu+u},x\right) - G\left(-\frac{1}{tu},0,1,1,x\right) + G\left(-\frac{1}{tu},0,1,-\frac{1}{t},x\right) + G\left(-\frac{1}{tu},0,1,-\frac{1-u}{tu+u},x\right) + \\
&G\left(-\frac{1}{tu},0,-\frac{1}{tu},1,x\right) - G\left(-\frac{1}{tu},0,-\frac{1}{tu},-\frac{1}{t},x\right) - G\left(-\frac{1}{tu},0,-\frac{1}{tu},-\frac{1-u}{tu+u},x\right) + G\left(-\frac{1}{tu},0,-\frac{1-u}{tu+u},1,x\right) - \\
&G\left(-\frac{1}{tu},0,-\frac{1-u}{tu+u},-\frac{1-u}{tu+u},x\right) + G\left(-\frac{1}{tu},1,0,-\frac{1}{t},x\right) + G\left(-\frac{1}{tu},1,1,1,x\right) - G\left(-\frac{1}{tu},1,1,-\frac{1-u}{tu+u},x\right) - G\left(-\frac{1}{tu},1,-\frac{1}{t},-\frac{1}{t},x\right) - \\
&G\left(-\frac{1}{tu},1,-\frac{1-u}{tu+u},1,x\right) + G\left(-\frac{1}{tu},1,-\frac{1-u}{tu+u},-\frac{1-u}{tu+u},x\right) - G\left(-\frac{1}{tu},-\frac{1}{t},0,1,x\right) + G\left(-\frac{1}{tu},-\frac{1}{t},0,-\frac{1-u}{tu+u},x\right) + \\
&G\left(-\frac{1}{tu},-\frac{1}{t},1,1,x\right) - G\left(-\frac{1}{tu},-\frac{1}{t},1,-\frac{1}{t},x\right) - G\left(-\frac{1}{tu},-\frac{1}{t},1,-\frac{1-u}{tu+u},x\right) - G\left(-\frac{1}{tu},-\frac{1}{t},-\frac{1}{tu},1,x\right) + \\
&G\left(-\frac{1}{tu},-\frac{1}{t},-\frac{1}{tu},-\frac{1}{t},x\right) + G\left(-\frac{1}{tu},-\frac{1}{t},-\frac{1}{tu},-\frac{1-u}{tu+u},x\right) + G\left(-\frac{1}{tu},-\frac{1}{tu},0,1,x\right) - G\left(-\frac{1}{tu},-\frac{1}{tu},0,-\frac{1}{t},x\right) - \\
&G\left(-\frac{1}{tu},-\frac{1}{tu},0,-\frac{1-u}{tu+u},x\right) - G\left(-\frac{1}{tu},-\frac{1}{tu},1,1,x\right) + G\left(-\frac{1}{tu},-\frac{1}{tu},1,-\frac{1-u}{tu+u},x\right) + G\left(-\frac{1}{tu},-\frac{1}{tu},-\frac{1}{t},-\frac{1}{t},x\right)
\end{aligned}
$$

The integrals

- When the smoke clears, one finds:



$$= \frac{160}{\epsilon^5} - \frac{1712}{\epsilon^4} + \frac{1}{\epsilon^3}\Big(- 120\,\zeta_2 + 2784\Big) + \frac{1}{\epsilon^2}\Big(- 120\,\zeta_3 + 1284\,\zeta_2 + 31968\Big)$$
$$+ \frac{1}{\epsilon}\Big(2520\,\zeta_4 + 1284\,\zeta_3 - 2088\,\zeta_2 - 216864\Big) + 15720\,\zeta_5 + 1920\,\zeta_2\,\zeta_3$$
$$- 26964\,\zeta_4 - 2088\,\zeta_3 - 23976\,\zeta_2 + 795744 + \epsilon\Big(82520\,\zeta_6 + 9600\,\zeta_3^2$$
$$- 168204\,\zeta_5 - 20544\,\zeta_2\,\zeta_3 + 43848\,\zeta_4 - 23976\,\zeta_3 + 162648\,\zeta_2 - 2449440\Big)$$
$$+ \mathcal{O}(\epsilon^2)\,.$$

- Thanks to these modern techniques we were able to compute all integrals analytically

- We obtain the soft-virtual approximation of the gluon fusion cross section at N3LO

- Why do we compute? ✔
- What do we want to compute? ✔
- How do we compute it? ✔
- What do we find?

The soft-virtual cross section at N3LO

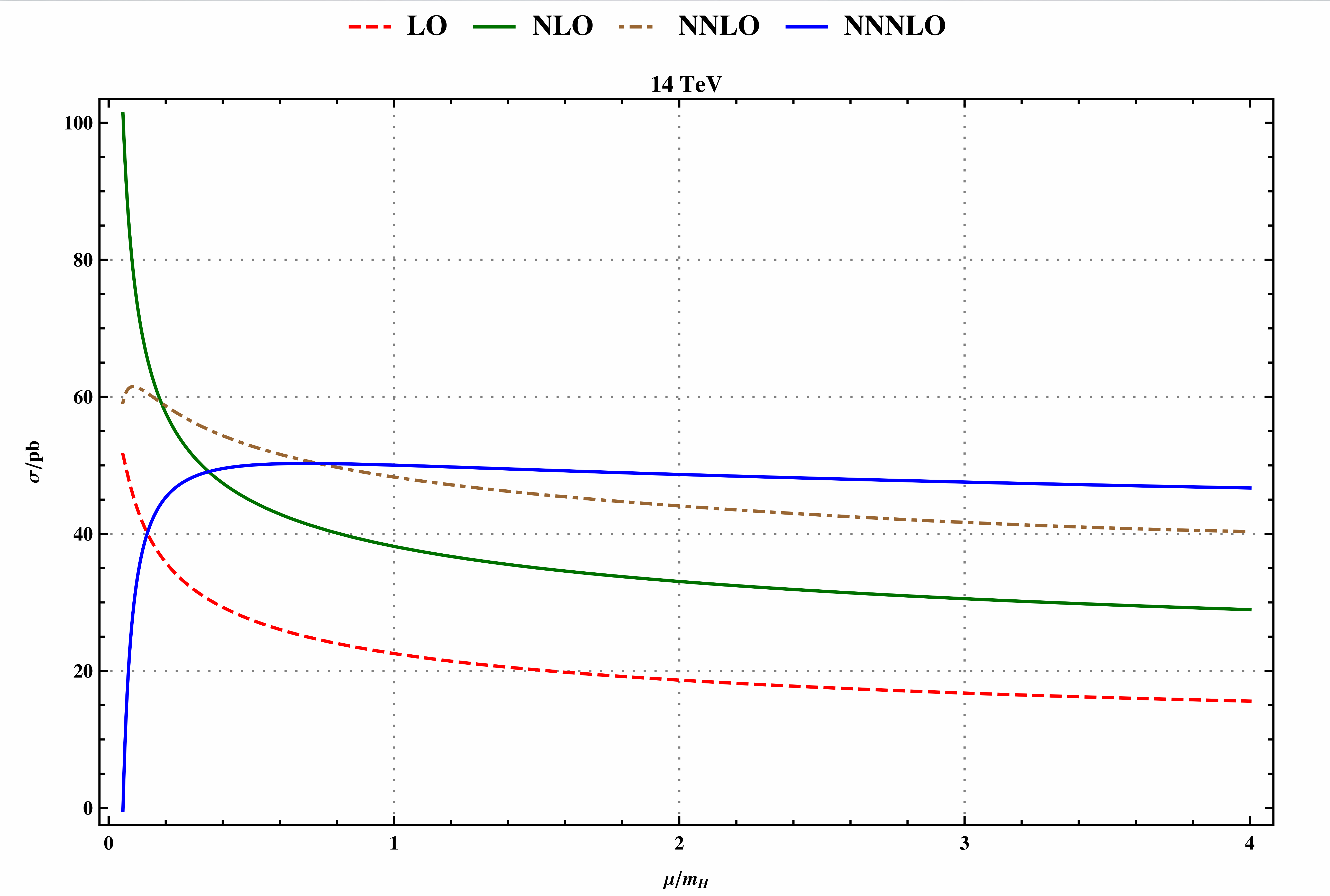
$$
\begin{aligned}
\hat{\eta}^{(3)}(z) = \delta(1-z)\Bigg\{ & C_A^3\left(-\frac{2003}{48}\zeta_6+\frac{413}{6}\zeta_3^2-\frac{7579}{144}\zeta_5+\frac{979}{24}\zeta_2\zeta_3-\frac{15257}{864}\zeta_4-\frac{819}{16}\zeta_3+\frac{16151}{1296}\zeta_2+\frac{215131}{5184}\right)\\
&+N_F\Bigg[C_A^2\left(\frac{869}{72}\zeta_5-\frac{125}{12}\zeta_3\zeta_2+\frac{2629}{432}\zeta_4+\frac{1231}{216}\zeta_3-\frac{70}{81}\zeta_2-\frac{98059}{5184}\right)\\
&\quad+C_A C_F\left(\frac{5}{2}\zeta_5+3\zeta_3\zeta_2+\frac{11}{72}\zeta_4+\frac{13}{2}\zeta_3-\frac{71}{36}\zeta_2-\frac{63991}{5184}\right)+C_F^2\left(-5\zeta_5+\frac{37}{12}\zeta_3+\frac{19}{18}\right)\Bigg]\\
&+N_F^2\Bigg[C_A\left(-\frac{19}{36}\zeta_4+\frac{43}{108}\zeta_3-\frac{133}{324}\zeta_2+\frac{2515}{1728}\right)+C_F\left(-\frac{1}{36}\zeta_4-\frac{7}{6}\zeta_3-\frac{23}{72}\zeta_2+\frac{4481}{2592}\right)\Bigg]\Bigg\}\\
+\left[\frac{1}{1-z}\right]_+\Bigg\{ & C_A^3\left(186\zeta_5-\frac{725}{6}\zeta_3\zeta_2+\frac{253}{24}\zeta_4+\frac{8941}{108}\zeta_3+\frac{8563}{324}\zeta_2-\frac{297029}{23328}\right)+N_F^2 C_A\left(\frac{5}{27}\zeta_3+\frac{10}{27}\zeta_2-\frac{58}{729}\right)\\
&+N_F\Bigg[C_A^2\left(-\frac{17}{12}\zeta_4-\frac{475}{36}\zeta_3-\frac{2173}{324}\zeta_2+\frac{31313}{11664}\right)+C_A C_F\left(-\frac{1}{2}\zeta_4-\frac{19}{18}\zeta_3-\frac{1}{2}\zeta_2+\frac{1711}{864}\right)\Bigg]\Bigg\}\\
+\left[\frac{\log(1-z)}{1-z}\right]_+\Bigg\{ & C_A^3\left(-77\zeta_4-\frac{352}{3}\zeta_3-\frac{152}{3}\zeta_2+\frac{30569}{648}\right)+N_F^2 C_A\left(-\frac{4}{9}\zeta_2+\frac{25}{81}\right)\\
&+N_F\Bigg[C_A^2\left(\frac{46}{3}\zeta_3+\frac{94}{9}\zeta_2-\frac{4211}{324}\right)+C_A C_F\left(6\zeta_3-\frac{63}{8}\right)\Bigg]\Bigg\}\\
+\left[\frac{\log^2(1-z)}{1-z}\right]_+ & \Bigg\{C_A^3\left(181\zeta_3+\frac{187}{3}\zeta_2-\frac{1051}{27}\right)+N_F\Bigg[C_A^2\left(-\frac{34}{3}\zeta_2+\frac{457}{54}\right)+\frac{1}{2}C_A C_F\Bigg]-\frac{10}{27}N_F^2 C_A\Bigg\}\\
+\left[\frac{\log^3(1-z)}{1-z}\right]_+ & \Bigg\{C_A^3\left(-56\zeta_2+\frac{925}{27}\right)-\frac{164}{27}N_F C_A^2+\frac{4}{27}N_F^2 C_A\Bigg\}\\
+\left[\frac{\log^4(1-z)}{1-z}\right]_+ & \left(\frac{20}{9}N_F C_A^2-\frac{110}{9}C_A^3\right)+\left[\frac{\log^5(1-z)}{1-z}\right]_+ 8C_A^3\,.
\end{aligned}
$$

[Anastasiou, Duhr, FD, Furlan, Gehrmann, Herzog, Mistlberger]

The soft-virtual cross section at N3LO

- This result contains the full three-correction and all corrections coming from the emission of up to three soft gluons
- How did we make sure that it is correct?
    - We observe the extremely intricate cancellation of six poles in dimensional regularisation
    - The plus distribution terms agree with a calculation by Moch and Vogt
    - All master integrals were calculated analytically and cross checked numerically
    - We performed internal independent calculations for all pieces and some contributions have been calculated and confirmed by other groups as well

The soft-virtual cross section at N3LO



The soft-virtual cross section at N3LO

- Caveat: The soft-virtual term alone is ambiguous

- Formally sub-leading terms could be inflated

$$\sigma = \int dx_1 dx_2 \mathrm{pdf}(x_1)\mathrm{pdf}(x_2)[zg(z)] \left[\frac{\hat{\sigma}(z)}{zg(z)}\right]_{\mathrm{threshold}}$$

- One can choose any $g(z)$ as long as $\lim_{z\to 1} g(z) = 1$

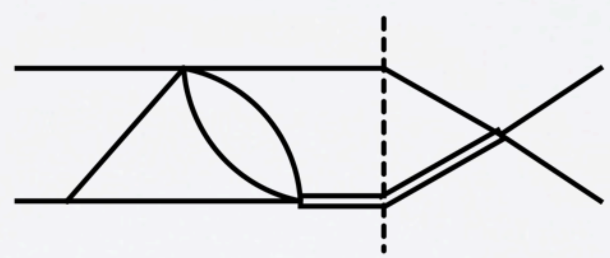
$g(z)$	1	z	z^2	$1/z$
$\frac{\delta\sigma^{N3LO}}{\sigma^{LO}}$	-2.27%	8.19%	30.16%	7.73%

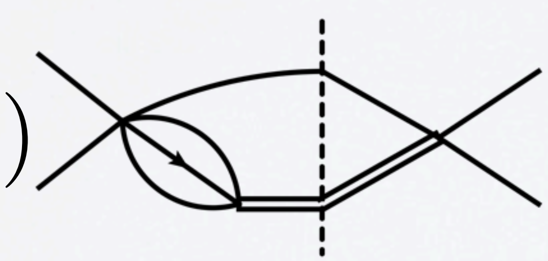
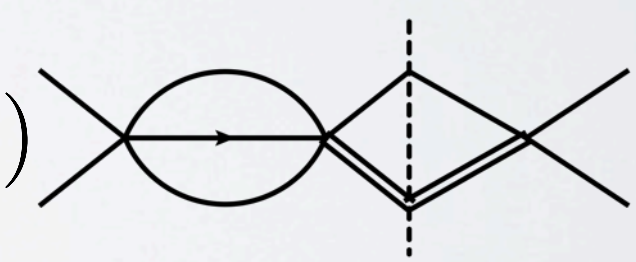
- More terms in the expansion / an unexpanded result are desirable

Outlook

- To obtain the full cross section we have to compute unexpanded master integrals

- Thanks to reverse unitarity we are able to derive differential equations for the master integrals

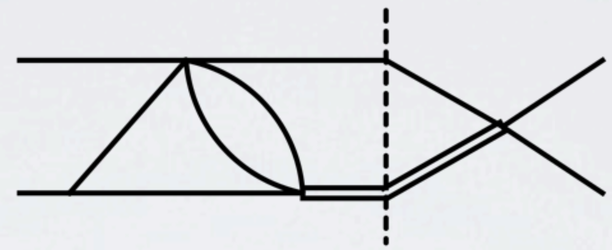
$\left[\partial_{\bar z} - 3\epsilon\ \mathrm{dlog}(1-\bar z)\right]$ 

$= \epsilon\ \mathrm{dlog}(1-\bar z)$  $-3\epsilon\ \mathrm{dlog}(1-\bar z)$ 

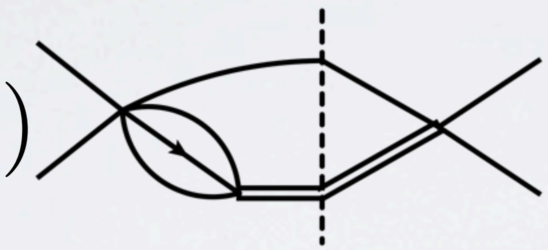
- Our calculation of the soft-virtual term provides us with many of the boundary conditions needed to solve the differential equations

Differential equations

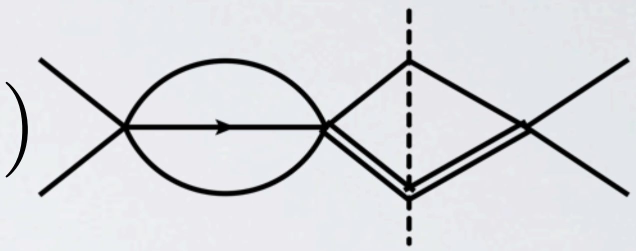
$$\left[\partial_{\bar{z}} - 3\epsilon \, \mathrm{dlog}(1 - \bar{z})\right]$$



$$= \epsilon \, \mathrm{dlog}(1 - \bar{z})$$



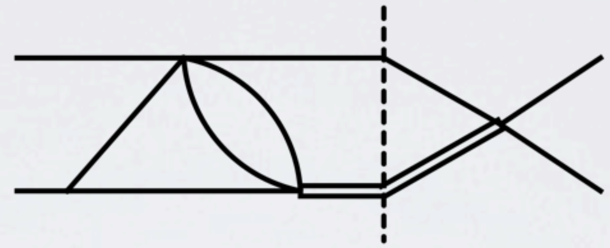
$$-3\epsilon \, \mathrm{dlog}(1 - \bar{z})$$



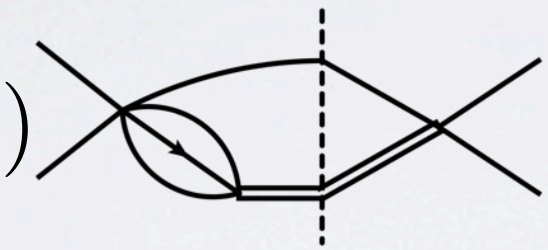
- In recent years there has been a lot of progress in bringing the differential equations into a simple form [Gehrmann, Remiddi; Henn]
- In this canonical form the differentials can be expressed as dlogs
- This makes the integration trivial
- Makes the singularity structure of the function obvious

Differential equations

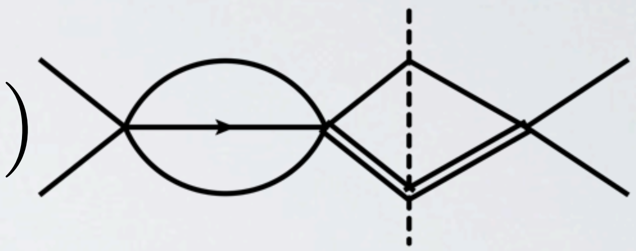
$$\left[\partial_{\bar{z}} - 3\epsilon \; \mathrm{dlog}(1-\bar{z})\right]$$



$$= \epsilon \; \mathrm{dlog}(1-\bar{z})$$



$$-3\epsilon \; \mathrm{dlog}(1-\bar{z})$$



- Compare with the definition of the multiple polylogarithms

$$G\left(a_1, \ldots, a_n; z\right) = \int_0^z \frac{dt}{t-a_1} G(a_2, \ldots, a_n; t)$$

- Polylogarithms are iterated integrals over dlog forms
- Trivial to write down the solution of the differential equation

Conclusion

- We computed the soft-virtual approximation of the gluon fusion cross section at N3LO
- The scale uncertainty is reduced to 4-5%
- Recent advances from number theory were required
- These modern techniques allowed us to compute integrals that would be impossible with conventional methods

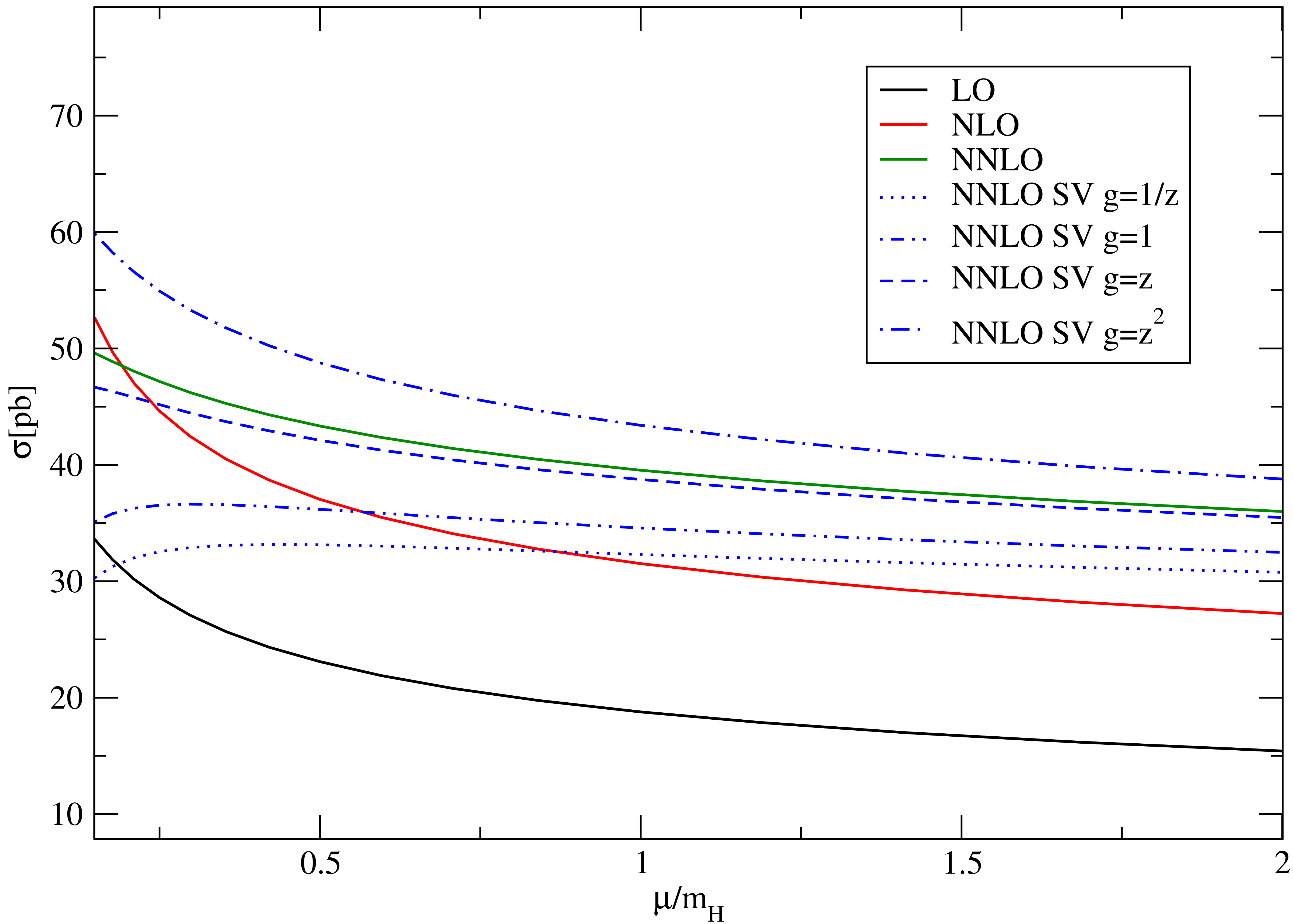
Conclusion

- The soft-virtual is a first approximation of the Higgs cross section at N3LO
- More terms in the expansion or a full calculation are needed for reliable phenomenology
- We expect to have more terms soon
- The full result is on the horizon
- The results obtained for the Higgs are easily transferable to other processes: Drell-Yan, SuSy Higgs, etc

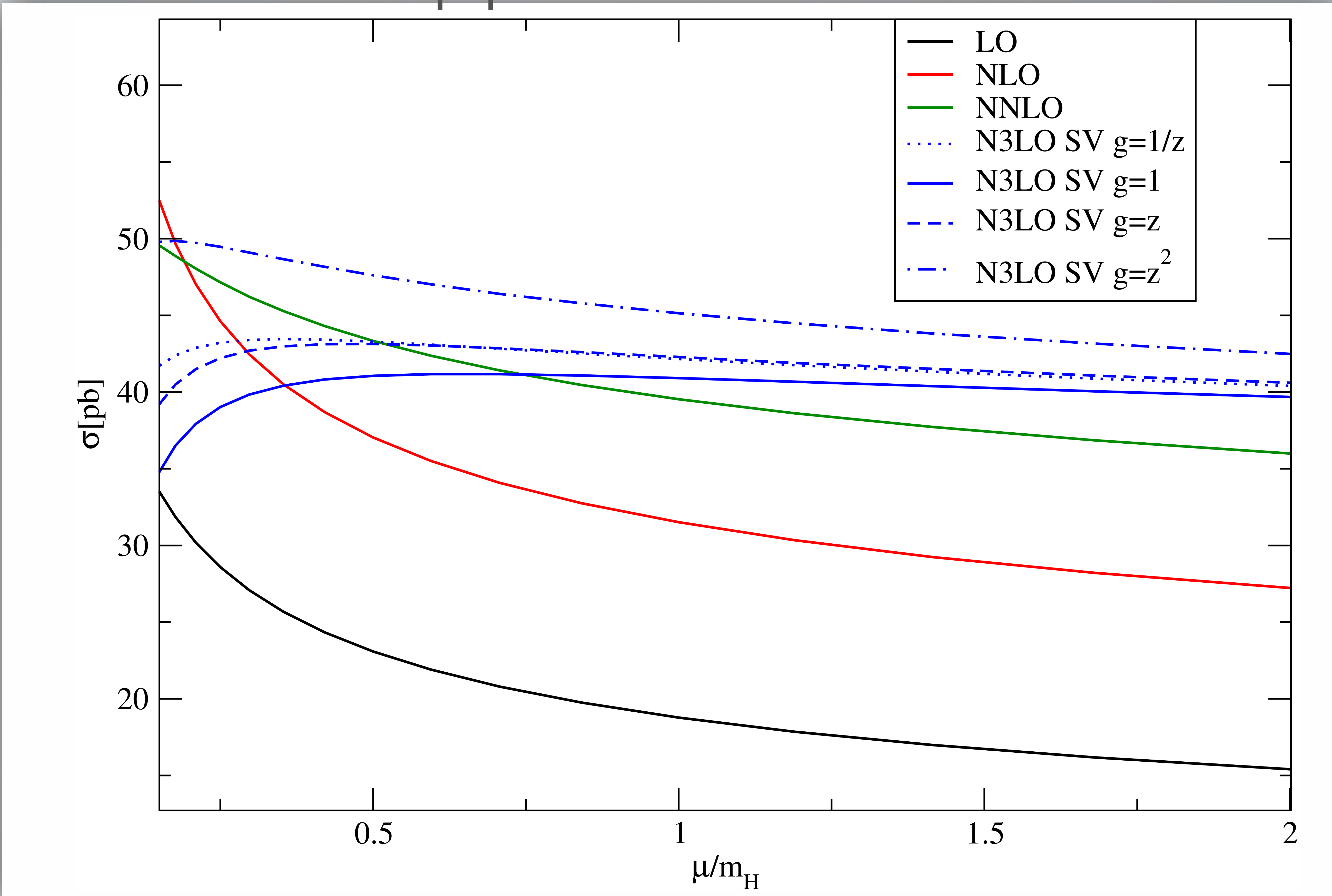
Thank you for your attention

Backup slides

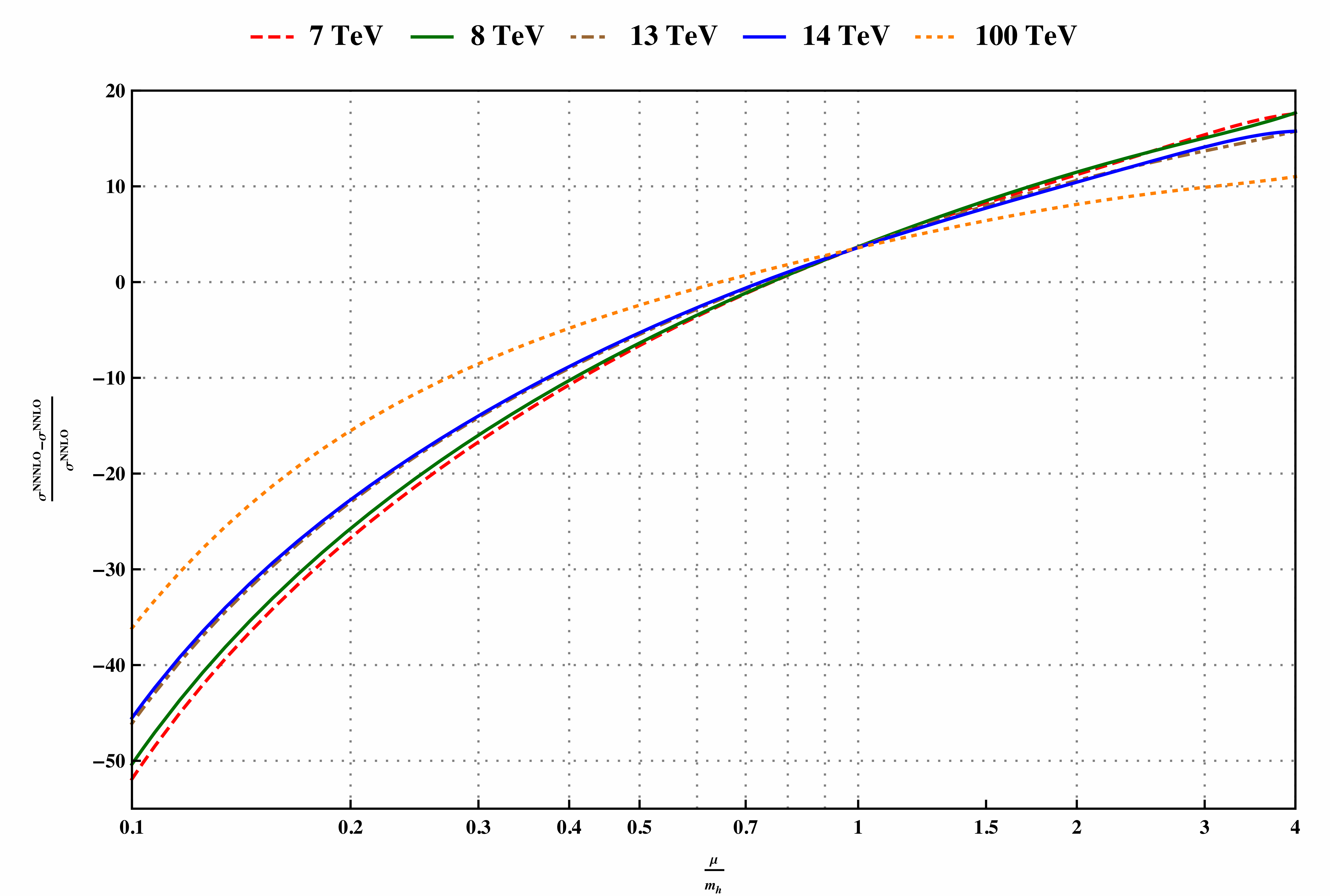
The soft approximation at NNLO



The soft approximation at N3LO

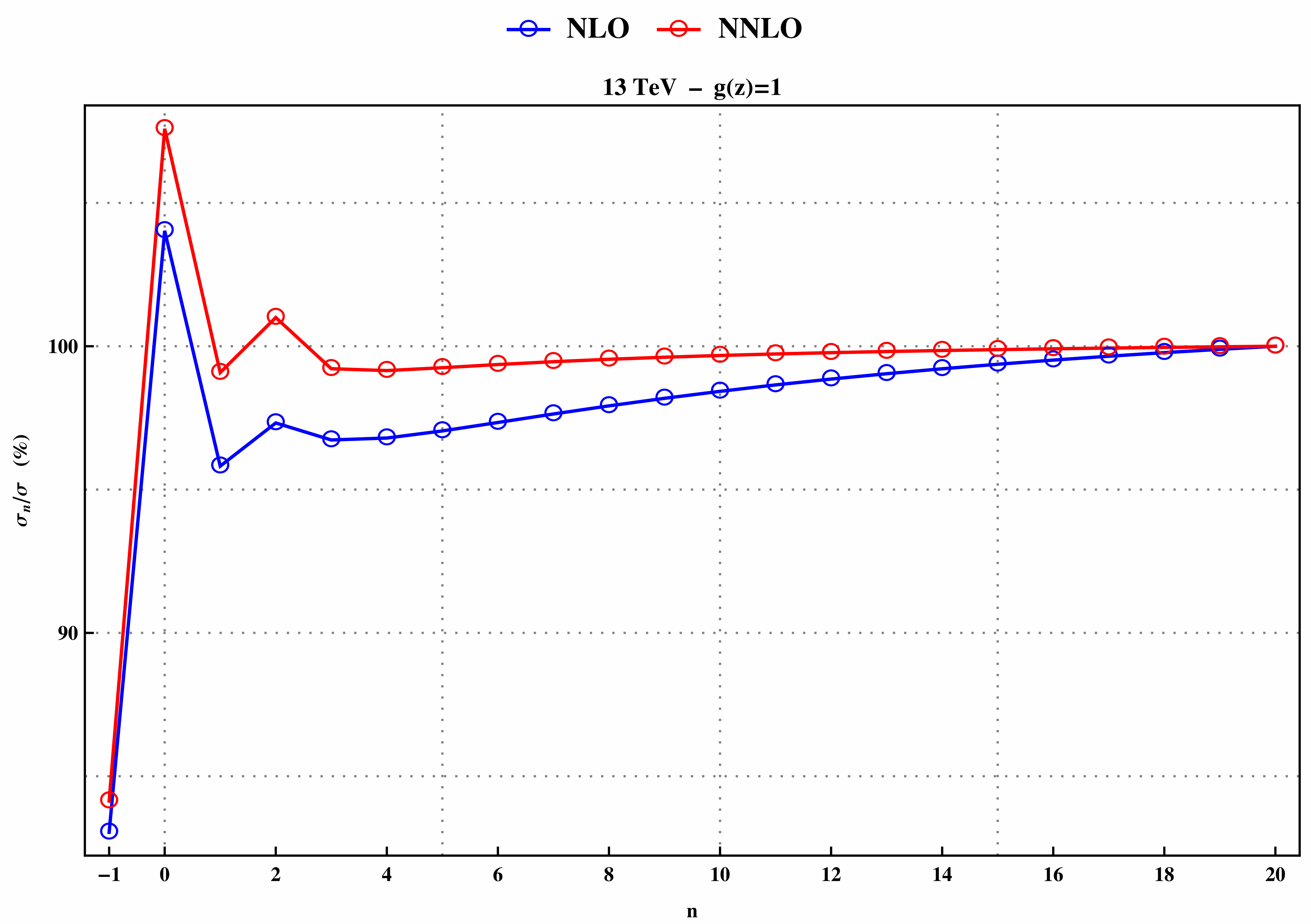


The soft-virtual cross section at N3LO



Convergence of the threshold expansion

How good is this approximation?



Convergence of the threshold expansion

How good is this approximation?